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Non-differentiable embedding of Lagrangian
systems and partial differential equations

by

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NON-DIFFERENTIABLE EMBEDDING OF LAGRANGIAN SYSTEMS AND PARTIAL DIFFERENTIAL EQUATIONS

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Au sujet du principe de moindre quantité d'action, je pense pour ma part qu'on peut le considérer comme la clé universelle de tous les problèmes, tant statiques que dynamiques, pour les questions relevant du mouvement des corps - quels que soit leur nombre et quelle que soit la manière dont ils sont liés entre eux - soit de l'équilibre et du mouvement des fluides quelconques...

Lettre de Lagrange à Euler du 17 Mai 1756

Abstract. — We develop the non-differentiable embedding theory of differential operators and Lagrangian systems using a new operator on non-differentiable functions. We then construct the corresponding calculus of variations and we derive the associated non-differentiable Euler-Lagrange equation, and apply this formalism to the study of PDE's. First, we extend the characteristics method to the non-differentiable case. We prove that non-differentiable characteristics for the Navier-Stokes equation correspond to extremals of an explicit non-differentiable Lagrangian system. Second, we prove that the solutions of the Schrödinger equation are non-differentiable extremals of the Newton's Lagrangian.

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Introduction

This paper is a contribution to the general program of embedding theories of dynamical systems [6]. Following our previous work on the stochastic embedding theory developed with Darses ([8],[9]) and the fractional embedding theory [7], we define the non-differentiable embedding of differential operators and equations using the quantum calculus [12].

The need for such a theory comes from Physics. In 1992, Nottale introduced the theory of scale-relativity without the hypothesis of space-time differentiability [14, 13]. As a consequence, at the microscopic scale one must take into account this loss of differentiability. Natural trajectories are everywhere non-differentiable. Nottale studies the effect of the loss of differentiability on classical equations of mechanics. More precisely, given an ordinary differential equation (ODE), Nottale's proposal can be mathematically translated as follow:

1. The ODE is the restriction over differentiable solutions of a more general "differential" equation on non-differentiable functions.
2. This extended equation is obtained by changing the classical derivative d/dt by a new operator acting on non-differentiable functions and recovering d/dt over differentiable ones.
3. If the initial equation has a specific structure, like being solution of a variational problem, one must recover the analogous of this structure in the non-differentiable context.

The last assumption is related to Nottale's interpretation of the relativity principle. The non-differentiable embedding gives a mathematical meaning to 1 and 2. For 3, we mainly focus on the non-differentiable embedding of the Euler-Lagrange equation which governs most of the dynamical behavior of physical systems in classical mechanics and physics [2]. The Euler-Lagrange equation comes from a variational principle called the least-action

principle which is one of the fundamental principle of Physics. The least-action principle is based on a functional called the action functional, also called the Lagrangian functional, which is completely determined by a scalar function called the Lagrangian. The mathematical tool to study these functionals is the classical calculus of variations.

Using the non-differentiable embedding we obtain a natural non-differentiable analogous of this equation. However, this result is by itself not sufficient. Indeed, as we have a non-differentiable analogue of Euler-Lagrange equation, we are lead to the following problems:

1. Develop a calculus of variations on non-differentiable functionals.
2. State the corresponding *non-differentiable least-action principle*, in particular explicit the associated *non-differentiable Euler-Lagrange equation* denoted by NDEL.
3. Compare the result with the non-differentiable embedded Euler-Lagrange equation $\text{Emb}(EL)$.

Following our previous work [5], we develop in this paper a non-differentiable calculus of variations and we obtain the corresponding non-differentiable least-action principle. We prove that the non-differentiable embedding is coherent, i.e. that we have $NDEL = \text{Emb}(EL)$. As a consequence, our construction takes into account the underlying variational nature of the Euler-Lagrange equation in the non-differentiable case.

We then provide two applications of this formalism. First, we extend the classical characteristics method for PDE's to the non-differentiable setting. As a consequence, we can consider PDE's of mixed type like the Navier-Stokes equation. We prove that the non-differentiable characteristic of the Navier-Stokes equation are non-differentiable extremals of an explicit Lagrangian functional. The non-differentiability is related to the viscosity coefficient. Second, following [5], we obtain a non-differentiable Lagrangian structure for the Schrödinger equation in arbitrary dimension.

The paper is organized as follow:

In part I, we first give some notations, and discuss about the irreversibility. Then, we define the non-differentiable embedding. Part II is devoted to the non-differentiable embedding of Lagrangian, while part III concerns the non-differentiable embedding of Hamiltonian systems. In both cases, we get a coherence principle for the Lagrangian system, meaning that the following diagram is commutative.

$$\begin{array}{ccc}
 \text{Lagrangian} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Lagrangian} \\
 \text{L.A.P.} \downarrow & & \downarrow \text{N.D.L.A.P.} \\
 \text{Euler-Lagrange equation} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Euler-Lagrange equation}
 \end{array}$$

In part IV, we discuss applications to PDE's and in particular the Schrödinger and the Navier-Stokes equations.

PART I

ABOUT NON-DIFFERENTIABLE EMBEDDING

1. Introduction

Classical equations of mechanics or physics are written using differential or partial differential equations. These equations are derived from experimental data measuring for example the successive positions x_i , $i = 1, \dots, n$, of a particle at times t_i , $i = 1, \dots, n$ respectively. Assuming that the particle moves continuously⁽¹⁾ along a path $x(t)$, $t \in \mathbb{R}$, we denote by $x(t_i) := x_i$. Usually one also computes the mean velocity v_i of the particle during the interval of time $\Delta t_i = t_i - t_{i-1}$ for $i = 2, \dots, n$. Using all these quantities we are able to construct a differential equation of the form

$$\frac{d_\epsilon^- x}{dt}(t) = v(x(t), t, \epsilon),$$

where v , the velocity, is determined by the experience and d_ϵ^- stands for the ϵ -mean left derivative, *i.e.*

$$\frac{d_\epsilon^- x}{dt}(a) := \frac{x(a) - x(a - \epsilon)}{\epsilon}.$$

Assuming that the quantity d_ϵ^- has a limit when ϵ goes to zero, we obtain a *backward differential equation*

$$\frac{d^- x}{dt}(t) = F(x(t), t),$$

where d^- is the left derivative of x , *i.e.*

$$\frac{d^- x}{dt}(a) := \lim_{t \rightarrow a^-} \frac{x(a) - x(t)}{a - t}.$$

In general one cannot produce the corresponding *forward differential equation*, because of the existence of an *arrow of time*. We only have access to a set of informations concerning the past of the particle. In order to use the data to construct the forward differential equation one makes a strong assumption namely the *reversibility* assumption, precisely it means that the arrow of time does not come into play. In that case, the forward differential equation can be deduced from the backward one and we obtain an ordinary differential equation

$$\frac{dx}{dt}(t) = F(x, t),$$

which encodes this assumption.

Rejecting the reversibility assumption we must deal with a set of two differential equations,

⁽¹⁾This assumption is of course far from being trivial in particular when dealing with quantum mechanics and is called the *assumption of continuity* by E. Schrödinger [15].

the forward and backward one, that describe completely the dynamical behavior of the particle, *i.e.*

$$(1) \quad \begin{cases} \frac{d^-x}{dt}(t) = F^-(x, t), \\ \frac{d^+x}{dt}(t) = F^+(x, t) \end{cases}$$

which leaves open the question of reversibility/irreversibility of the underlying process.

One can introduce a new operator taking into account d^\pm using complex numbers. Indeed, we define a complex valued operator denoted by D_μ and defined by

$$D_\mu := \frac{d^+ + d^-}{2} + i\mu \frac{d^+ - d^-}{2},$$

where $\mu \in \mathbb{C}$ is only of use to recover special cases of interest, *i.e.* for $\mu = i$, $D_i = d^-$ and for $\mu = -i$, $D_{-i} = d^+$.

Remark 1. — *We can go further by taking two independent variables x^+ and x^- for the forward and backward time evolution of the physical system. This will be explored in a forthcoming paper. In the case we need to weight the dynamical information coming from the past or the future, we could use the fractional version of this construction. We refer to [7] for more details.*

Our discussion about reversibility/irreversibility justifies the introduction of the left and right derivatives. In order to cover non-differentiable curves we consider a weaker notion of derivatives given by the quantum calculus [12].

2. Reminder about the quantum calculus

In this section, we recall the definition as well as the basic properties of the quantum calculus and the ϵ -scale derivatives introduced in ([4] and [5], see also [12]).

2.1. Notations. — Let $d \in \mathbb{N}$ be a fixed integer, I an open set in \mathbb{R} , and $a, b \in \mathbb{R}$, $a < b$, such that $[a, b] \subset I$, be given in the whole paper. We denote by $\mathcal{F}(I, \mathbb{R}^d)$ the set of functions $x : I \rightarrow \mathbb{R}^d$ from I to \mathbb{R}^d , and $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) the subset of $\mathcal{F}(I, \mathbb{R}^d)$ (respectively $\mathcal{F}(I, \mathbb{C}^d)$) which are continuous. Let $n \in \mathbb{N}$, we denote by $\mathcal{C}^n(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^n(I, \mathbb{C}^d)$) the set of functions in $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) which are differentiable up to order n .

Definition 1. — (*Hölderian functions*) Let $w \in \mathcal{C}^0(I, \mathbb{R}^d)$. Let $t \in I$.

1. w is Hölder of Hölder exponent α , $0 < \alpha < 1$, at point t if

$$\exists c > 0, \exists \eta > 0 \text{ s.t. } \forall t' \in I \mid |t - t'| \leq \eta \Rightarrow \|w(t) - w(t')\| \leq c |t - t'|^\alpha,$$

where $\|\cdot\|$ is a norm on \mathbb{R}^d .

2. w is α -Hölder and inverse Hölder with $0 < \alpha < 1$, at point t if

$$\begin{aligned} \exists c, C \in \mathbb{R}^{+*}, c < C, \exists \eta > 0 \text{ s.t. } \forall t' \in I \mid |t - t'| \leq \eta \\ c |t - t'|^\alpha \leq \|w(t) - w(t')\| \leq C |t - t'|^\alpha. \end{aligned}$$

A complex valued function is α -Hölder if its real and imaginary parts are α -Hölder. We denote by $H^\alpha(I, \mathbb{R}^d)$ the set of continuous functions α -Hölder and by $\mathbb{H}^\alpha(I, \mathbb{R}^d)$ the set of continuous functions α -Hölder and α -inverse Hölder. For explicit examples of α -Hölder and α -inverse Hölder functions we refer to ([16], p.168) in particular the Knopp or Takagi function.

Definition 2. — Let $\mathcal{C}^{k,\alpha}(I, \mathbb{C}^d)$ be the subset of $\mathcal{C}^0(I, \mathbb{C}^d)$ of functions $x : I \rightarrow \mathbb{C}^d$ of the form

$$x(t) := u(t) + w(t),$$

for any $t \in I$, where $u \in \mathcal{C}^k(I, \mathbb{C}^d)$ and $w \in \mathbb{H}^\alpha(I, \mathbb{C}^d)$.

2.2. Quantum calculus. — For a general continuous function, we cannot define the derivative at a given point. However, for all $\epsilon > 0$, we have access to the left and right quantum derivatives, which are the *discrete* versions of the left and right derivatives.

Definition 3. — Let $x \in \mathcal{C}^0(I, \mathbb{R}^d)$. For all $\epsilon > 0$, we call ϵ -left and right quantum derivatives the quantities $d_\epsilon^\sigma x$ defined for any $t \in I$ by

$$d_\epsilon^\sigma x(t) := \sigma \frac{x(t + \sigma\epsilon) - x(t)}{\epsilon}, \quad \sigma = \pm.$$

The ϵ -left and right quantum derivatives of a continuous function correspond to the classical derivatives of the left and right ϵ -mean function defined by

$$x_\epsilon^\sigma(t) := \frac{\sigma}{\epsilon} \int_t^{t+\sigma\epsilon} x(s) ds, \quad \sigma = \pm.$$

Then, d_ϵ^σ can be interpreted as the left and right derivatives at a given scale ϵ .

Using ϵ -left and right derivatives, we can define the quantum derivative, $\frac{\square_\epsilon x}{\square t}$, which generalizes the classical derivative.

Definition 4. — Let $x \in \mathcal{C}^0(I, \mathbb{R}^d)$. For any $\epsilon > 0$, the ϵ -scale derivative of x is the quantity denoted by $\frac{\square_\epsilon x}{\square t} : \mathcal{C}^0(I, \mathbb{R}^d) \rightarrow \mathcal{C}^0(I, \mathbb{C}^d)$, and defined at point t by

$$\frac{\square_\epsilon x}{\square t}(t) := \frac{1}{2} \left[(d_\epsilon^+ x(t) + d_\epsilon^- x(t)) + i\mu (d_\epsilon^+ x(t) - d_\epsilon^- x(t)) \right],$$

where $\mu \in \{1, -1, 0, i, -i\}$.

If x is differentiable, taking the limit of the ϵ -scale derivative when ϵ goes to zero, leads to $\frac{dx}{dt}$ the classical derivative of x . We will frequently denote $\square_\epsilon x$ for $\frac{\square_\epsilon x}{\square t}$. Moreover, we do not write the dependence of \square_ϵ on μ . Let us notice that for $\mu = i$ we obtain d_ϵ^- and for $\mu = -i$ we get d_ϵ^+ , which allows us to recover the backward and forward derivatives.

We also need to extend the scale derivative to complex valued functions in order to be able to compute *composition* of \square_ϵ . Indeed, if x is a real valued function $\square_\epsilon x$ is a complex valued function, and it is not clear what is the correct definition of $\square_\epsilon(\square_\epsilon x)$. Then, we choose to define \square_ϵ over complex valued functions as follows:

Definition 5. — Let $x \in \mathcal{C}^0(I, \mathbb{C}^d)$ be a continuous complex valued function. For all $\epsilon > 0$, the ϵ -scale derivative of x , denoted by $\frac{\square_\epsilon x}{\square t}$ is defined by

$$(2) \quad \frac{\square_\epsilon x}{\square t} := \frac{\square_\epsilon \operatorname{Re}(x)}{\square t} + i \frac{\square_\epsilon \operatorname{Im}(x)}{\square t},$$

where $\operatorname{Re}(x)$ and $\operatorname{Im}(x)$ are the real and imaginary part of x .

This extension of the quantum derivative in order to cover complex valued functions is far from being trivial. Indeed, it mixes complex terms in a complex operator. From an algebraic point of view it means the operator \square_ϵ has to be \mathbb{C} -linear (see also [9]).

Let $\mathcal{C}_{conv}^0(I \times]0, 1], \mathbb{R}^d)$ be a sub-vectorial space of $\mathcal{C}^0(I \times]0, 1], \mathbb{R}^d)$ such that for any function $f \in \mathcal{C}_{conv}^0(I \times]0, 1], \mathbb{R}^d)$ the limit $\lim_{\epsilon \rightarrow 0} f(t, \epsilon)$ exists for any $t \in I$. We denote by E a complementary space of $\mathcal{C}_{conv}^0(I \times]0, 1], \mathbb{R}^d)$ in $\mathcal{C}^0(I \times]0, 1], \mathbb{R}^d)$ and by π the projection onto $\mathcal{C}_{conv}^0(I \times]0, 1], \mathbb{R}^d)$ by

$$\begin{aligned} \pi : \mathcal{C}_{conv}^0(I \times]0, 1], \mathbb{R}^d) \oplus E &\rightarrow \mathcal{C}_{conv}^0(I \times]0, 1], \mathbb{R}^d) \\ f_{conv} + f_E &\mapsto f_{conv}. \end{aligned}$$

We can then define the operator $\langle . \rangle$ by

$$\begin{aligned} \langle . \rangle : \mathcal{C}^0(I \times]0, 1], \mathbb{R}^d) &\rightarrow \mathcal{F}(I, \mathbb{R}^d) \\ f &\mapsto \langle \pi(f) \rangle : t \mapsto \lim_{\epsilon \rightarrow 0} \pi(f)(t, \epsilon). \end{aligned}$$

Definition 6. — Let us introduce the new operator $\frac{\square}{\square t}$ (without ϵ) on the space $\mathcal{C}^0(I, \mathbb{R}^d)$ by:

$$(3) \quad \frac{\square x}{\square t} := \langle \pi\left(\frac{\square_\epsilon x}{\square t}\right) \rangle$$

The operator $\frac{\square}{\square t}$ acts on complex valued functions by \mathbb{C} -linearity. For a differentiable function $x \in \mathcal{C}^1(I, \mathbb{R}^d)$, $\frac{\square x}{\square t} = \frac{dx}{dt}$, which is the classical derivative. More generally if $\frac{\square^k x}{\square t^k}$ denotes $\frac{\square^k x}{\square t^k} := \frac{\square}{\square t} \circ \dots \circ \frac{\square}{\square t} x$ and $x \in \mathcal{C}^k(I, \mathbb{R}^d)$, $k \in \mathbb{N}$, then $\frac{\square^k x}{\square t^k} = \frac{d^k x}{dt^k}$.

Note that if $x \in \mathcal{C}^{k,\alpha}(I, \mathbb{C}^d)$, $0 < \alpha < 1$, with $x := u + w$ (as in definition 2), we have

$$\frac{\square x(t)}{\square t} = u'(t).$$

Indeed, since $\|\square_\epsilon w(t)\| \geq c \cdot \epsilon^{\alpha-1}$, we obtain $\frac{\square w(t)}{\square t} = 0$.

Remark 2. — In [5] we introduce a bracket $[\cdot]_\epsilon$ which keeps the divergent information. It was defined as:

$$[\square_\epsilon x(t)]_\epsilon := \square x + x_E(t, \epsilon),$$

where x_E denotes the projection of x on E . As a consequence, we obtain more complicated quantities and in particular we keep a dependence on ϵ .

2.3. Non-differentiable Leibniz formula. —

Lemma 1. — Let $f \in H^\alpha(I, \mathbb{R}^d)$ and $g \in H^\beta(I, \mathbb{R}^d)$, with $\alpha + \beta > 1$,

$$(4) \quad \frac{\square}{\square t}(f \cdot g) = \frac{\square f}{\square t} \cdot g + f \cdot \frac{\square g}{\square t}.$$

Proof. — Let us first consider real valued functions $f \in H^\alpha(I, \mathbb{R})$ and $g \in H^\beta(I, \mathbb{R})$, and start with d_ϵ^+ , then

$$d_\epsilon^+(fg) = d_\epsilon^+ f g + f d_\epsilon^+ g + \epsilon d_\epsilon^+ f d_\epsilon^+ g.$$

Since f and g are respectively α and β -Hölder, we have $|d_\epsilon^+ f| \leq c_f \epsilon^{\alpha-1}$ and $|d_\epsilon^+ g| \leq c_g \epsilon^{\beta-1}$, then $|\epsilon d_\epsilon^+ f d_\epsilon^+ g| \leq c_f c_g \epsilon^{\alpha+\beta-1}$. This quantity converge to 0, when ϵ goes to 0, since $\alpha + \beta > 1$, so that $\langle \pi(\epsilon d_\epsilon^+(fg)) \rangle = 0$. The same holds for d_ϵ^- , we obtain

$$\langle \pi(d_\epsilon^\sigma(fg)) \rangle = \langle \pi(d_\epsilon^\sigma f) \rangle g + f \langle \pi(d_\epsilon^\sigma g) \rangle.$$

By linearity, we get $\frac{\square}{\square t}(fg) = \frac{\square f}{\square t} \cdot g + f \cdot \frac{\square g}{\square t}$ for real valued functions. The generalization to vector valued functions $f \in H^\alpha(I, \mathbb{R}^d)$ and $g \in H^\beta(I, \mathbb{R}^d)$ is straightforward. \square

Definition 7. — We denote by $\mathcal{C}_\square^1(I, \mathbb{R})$ the set of continuous functions $f \in \mathcal{C}^0(I, \mathbb{R})$ such that $\frac{\square f}{\square t} \in \mathcal{C}^0(I, \mathbb{R})$.

Lemma 2. — Let $f \in \mathcal{C}_\square^1(I, \mathbb{R})$, then

$$(5) \quad \int_a^b \frac{\square f(t)}{\square t} dt = f(b) - f(a).$$

Proof. — With the definition of d_ϵ^σ , it is easy to check that $\lim_{\epsilon \rightarrow 0} \int_a^b d_\epsilon^\sigma f(t) dt = f(b) - f(a)$, $\sigma = \pm$. We deduce that

$$\lim_{\epsilon \rightarrow 0} \int_a^b \square_\epsilon f(t) dt = f(b) - f(a) \quad \text{and then} \quad \pi\left(\int_a^b \square_\epsilon f(t) dt\right) = \int_a^b \square_\epsilon f(t) dt.$$

On the other hand, using the definition of the operator π and the mean-value theorem for the divergent part, we obtain that

$$\int_a^b \pi(\square_\epsilon f(t)) dt = \pi\left(\int_a^b \square_\epsilon f(t) dt\right).$$

Passing to the limit when ϵ goes to zero, and using the definition of \square , we can conclude. \square

For $x \in \mathcal{C}^1(I, \mathbb{R})$ and $f \in \mathcal{C}^1(I, \mathbb{R})$, the classical differential calculus gives

$$\frac{df(x(t), t)}{dt} = \frac{\partial f}{\partial t}(x(t), t) + \frac{\partial f}{\partial x}(x(t), t) \cdot x'(t).$$

The analogous of the derivative of a composed function for the quantum derivative is given by the following theorem:

Theorem 1. — *Let f be a $\mathcal{C}^2(\mathbb{R}^d \times I, \mathbb{R})$ function and $x \in H^\alpha(I, \mathbb{R}^d)$, $\frac{1}{2} \leq \alpha < 1$, we have*

$$(6) \quad \frac{\square f(x(t), t)}{\square t} = \nabla_x f(x(t), t) \cdot \nabla_{\square} x(t) + \frac{\partial f}{\partial t}(x(t), t) + \sum_{k=1}^d \sum_{l=1}^d \frac{1}{2} \frac{\partial^2 f}{\partial x_k \partial x_l}(x(t), t) a_{k,l}(x(t))$$

with the notations

$$\nabla_{\square} x(t) := \left(\frac{\square x_1}{\square t}(t), \dots, \frac{\square x_d}{\square t}(t) \right)^t,$$

and

$$(7) \quad a_{k,l}(x(t)) := \left\langle \pi \left(\frac{\epsilon}{2} \left((d_\epsilon^+ x_k(t))(d_\epsilon^+ x_l(t))(1 + i\mu) - (d_\epsilon^- x_k(t))(d_\epsilon^- x_l(t))(1 - i\mu) \right) \right) \right\rangle$$

where i is the complex number.

Note that $a_{k,l}(x) = 0$ if $x \in \mathcal{C}^1(I, \mathbb{R}^d)$.

Proof. — Let us first consider d_ϵ^+ :

$$d_\epsilon^+(f(x(t), t)) = \frac{f(x(t + \epsilon), t + \epsilon) - f(x(t), t)}{\epsilon}.$$

Moreover

$$f(x(t + \epsilon), t + \epsilon) = f(\epsilon d_\epsilon^+(x(t)) + x(t), t + \epsilon).$$

As x is α -Hölder, $\|d_\epsilon^+(x)\| \leq c\epsilon^{\alpha-1}$, and $\epsilon\|d_\epsilon^+(x)\| \leq c\epsilon^\alpha$, $\lim_{\epsilon \rightarrow 0} \epsilon\|d_\epsilon^+(x)\| = 0$. Since f is of class \mathcal{C}^2 , we can consider its expansion:

$$\begin{aligned} f(x(t + \epsilon), t + \epsilon) &= f(x(t), t) + \sum_{k=1}^d \frac{\partial f}{\partial x_k}(x(t), t) (\epsilon d_\epsilon^+(x_k(t))) + \frac{\partial f}{\partial t}(x(t), t) \epsilon \\ &\quad + \sum_{k=1}^d \sum_{l=1}^d \frac{1}{2!} \frac{\partial^2 f}{\partial x_k \partial x_l}(x(t), t) (\epsilon d_\epsilon^+(x_k(t))) (\epsilon d_\epsilon^+(x_l(t))) \\ &\quad + \frac{1}{2!} \frac{\partial^2 f}{\partial t^2}(x(t), t) \epsilon^2 + \sum_{k=1}^d \frac{\partial^2 f}{\partial x_k \partial t}(x(t), t) (\epsilon d_\epsilon^+(x_k(t))) \epsilon \\ &\quad + o(\|(\epsilon d_\epsilon^+(x_1(t)), \dots, \epsilon d_\epsilon^+(x_d(t)), \epsilon)\|^2). \end{aligned}$$

Since $\|\epsilon d_\epsilon^+ x_k(t)\| \leq c_k \epsilon^\alpha$, for any $k = 1, \dots, d$, and $\frac{1}{2} \leq \alpha < 1$, we get

$$o(\|(\epsilon d_\epsilon^+(x_1(t)), \dots, \epsilon d_\epsilon^+(x_d(t)), \epsilon)\|^2) \leq \sum_{k=1}^d c_k^2 \epsilon^{2\alpha} + \epsilon^2 < c \epsilon^{2\alpha}$$

We obtain the following formula for d_ϵ^+ :

$$(8) \quad d_\epsilon^+(f(x(t), t)) = \sum_{k=1}^d \frac{\partial f}{\partial x_k}(x(t), t) d_\epsilon^+(x_k(t)) + \frac{\partial f}{\partial t}(x(t), t) \\ + \sum_{k=1}^d \sum_{l=1}^d \frac{1}{2!} \frac{\partial^2 f}{\partial x_k \partial x_l}(x(t), t) \epsilon d_\epsilon^+(x_k(t)) d_\epsilon^+(x_l(t)) + o(\epsilon^{2\alpha-1}).$$

Doing the same calculation for $d_\epsilon^- f$ gives:

$$(9) \quad d_\epsilon^-(f(x(t), t)) = \sum_{k=1}^d \frac{\partial f}{\partial x_k}(x(t), t) d_\epsilon^-(x_k(t)) + \frac{\partial f}{\partial t}(x(t), t) \\ - \sum_{k=1}^d \sum_{l=1}^d \frac{1}{2!} \frac{\partial^2 f}{\partial x_k \partial x_l}(x(t), t) \epsilon d_\epsilon^-(x_k(t)) d_\epsilon^-(x_l(t)) + o(\epsilon^{2\alpha-1}).$$

Finally, combining (8) and (9) leads to the result. \square

From the previous result we deduce the following formula for functions of $\mathcal{C}^{1,\alpha}$:

Corollary 1. — *Let f be a $\mathcal{C}^2(\mathbb{R}^d \times I, \mathbb{R})$ function and $\frac{1}{2} \leq \alpha < 1$. Let $x = (x_1, \dots, x_d) \in \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$ written as $x := u + w$ where $u = (u_1, \dots, u_d) \in \mathcal{C}^1(I, \mathbb{R}^d)$ and $w = (w_1, \dots, w_d) \in \mathbb{H}^\alpha(I, \mathbb{R}^d)$, then the following formula holds*

$$(10) \quad \frac{\square f(x(t), t)}{\square t} = \nabla_x f(x(t), t) \cdot \nabla u(t) + \frac{\partial f}{\partial t}(x(t), t) \\ + \sum_{k=1}^d \sum_{l=1}^d \frac{1}{2} \frac{\partial^2 f}{\partial x_k \partial x_l}(x(t), t) a_{k,l}(w(t)),$$

where the definition of $a_{k,l}$ is given in the previous lemma by (7).

In comparison with the classical derivative $\frac{d}{dt} f(x(t), t)$, the non-regular part of $x \in \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$ gives rise to the last term of the previous formula.

Proof. — Since $x := u + w$ where $u = (u_1, \dots, u_d) \in \mathcal{C}^1(I, \mathbb{R}^d)$ and $w = (w_1, \dots, w_d) \in \mathbb{H}^\alpha(I, \mathbb{R}^d)$, then

$$\lim_{\epsilon \rightarrow 0} \frac{\square_\epsilon u}{\square t}(t) = \nabla u(t) \quad \text{and} \quad \|d_\epsilon^+ w_k\| \sim \epsilon^{\alpha-1}, \quad \text{and} \quad \|d_\epsilon^- w_k\| \sim \epsilon^{\alpha-1}, \quad k = 1, \dots, d.$$

This implies

$$\frac{\square u}{\square t}(t) = \nabla u(t) \quad \text{and} \quad \frac{\square w}{\square t}(t) = 0.$$

Moreover

$$\lim_{\epsilon \rightarrow 0} ((d_\epsilon^+ u_k(t))(d_\epsilon^+ u_l(t))(1 + i\mu) - (d_\epsilon^- u_k(t))(d_\epsilon^- u_l(t))(1 - i\mu)) = 2i\mu u_k'(t) u_l(t)$$

so that $a_{k,l}(u(t)) = 0$, and $a_{k,l}(u(t)) = a_{k,l}(w(t))$ on the other side

$$\frac{\epsilon}{2}((d_\epsilon^+ w_k(t))(d_\epsilon^+ w_l(t))(1 + i\mu) - (d_\epsilon^- w_k(t))(d_\epsilon^- w_l(t))(1 - i\mu)) \sim \epsilon^{2\alpha-1}$$

□

3. Non-differentiable embedding

This section follows the strategy of the stochastic and fractional embeddings of differential operators (see [7] and [9]).

3.1. Non-differentiable embedding of differential operators. — Let $f : I \times \mathbb{C}^d \rightarrow \mathbb{C}$ be a function, real valued on real arguments. We denote by F the corresponding operator acting on functions x and defined by

$$F : \begin{array}{l} \mathcal{C}^0(I, \mathbb{C}^d) \\ x \end{array} \begin{array}{l} \longrightarrow \\ \longmapsto \end{array} \begin{array}{l} \mathcal{C}^0(I, \mathbb{C}) \\ f(\bullet, x(\bullet)) \end{array}$$

where $f(\bullet, x(\bullet))$ is the function defined by

$$f(\bullet, x(\bullet)) : \begin{array}{l} I \\ t \end{array} \begin{array}{l} \longrightarrow \\ \longmapsto \end{array} \begin{array}{l} \mathbb{C}, \\ f(t, x(t)). \end{array}$$

Let $\mathbf{f} = \{f_i\}_{i=0,\dots,n}$ (resp. $\mathbf{g} = \{g_i\}_{i=0,\dots,n}$) be a finite family of functions of class \mathcal{C}^n , $f_i : I \times \mathbb{C}^d \rightarrow \mathbb{C}$ (resp. $g_i : I \times \mathbb{C}^d \rightarrow \mathbb{C}$), and F_i (resp. G_i), $i = 0, \dots, n$ the corresponding family of operators.

Definition 8. — We denote by $O_{\mathbf{f}}^{\mathbf{g}}$ the differential operator acting on $\mathcal{C}^n(I, \mathbb{C}^d)$ defined by

$$(11) \quad O_{\mathbf{f}}^{\mathbf{g}} = \sum_{i=0}^n F_i \cdot \left(\frac{d^i}{dt^i} \circ G_i \right),$$

where \cdot is the standard product of operators, i.e. if A and B are two operators, we denote by $A \cdot B$ the operator defined by $(A \cdot B)(x) = A(x)B(x)$ and \circ the usual composition, i.e. $(A \circ B)(x) = A(B(x))$, with the convention that $\left(\frac{d}{dt} \right)^0 = \text{Id}$, where Id denotes the identity mapping on \mathbb{C} .

Definition 9 (Non-differentiable embedding of operators)

The non-differentiable embedding of $O_{\mathbf{f}}^{\mathbf{g}}$ written as (11), denoted by $\text{Emb}_{\square}(O_{\mathbf{f}}^{\mathbf{g}})$ is the operator

$$(12) \quad \text{Emb}_{\square}(O_{\mathbf{f}}^{\mathbf{g}}) = \sum_{i=0}^n F_i \cdot \left(\frac{\square^i}{\square t^i} \circ G_i \right).$$

Note that the embedding procedure acts on operators of a given form and not on operators like abstract data, i.e. this is not a mapping on the set of operators.

3.2. Non-differentiable embedding of differential equations. — Let $k \in \mathbb{N}$ be a fixed integer. Let $\mathbf{f} = \{f_i\}_{i=0,\dots,n}$ and $\mathbf{g} = \{g_i\}_{i=0,\dots,n}$ be finite families of functions of class \mathcal{C}^n , $f_i : \mathbb{R} \times \mathbb{C}^{kd} \rightarrow \mathbb{C}$ and $g_i : \mathbb{R} \times \mathbb{C}^{kd} \rightarrow \mathbb{C}$ respectively, and $F_i, G_i, i = 0, \dots, n$ the corresponding families of operators. We denote by $O_{\mathbf{f}}^{\mathbf{g}}$ the operator acting on $(\mathcal{C}^{k+n}(I, \mathbb{C}^d))^{k+1}$ defined by

$$O_{\mathbf{f}}^{\mathbf{g}} = \sum_{i=0}^n F_i \cdot \left(\frac{d^i}{dt^i} \circ G_i \right).$$

Definition 10. — Let the ordinary differential equation associated to $O_{\mathbf{f}}^{\mathbf{g}}$ be defined by

$$(13) \quad O_{\mathbf{f}}^{\mathbf{g}} \left(x, \frac{dx}{dt}, \dots, \frac{d^k x}{dt^k} \right) = 0, \quad \text{for any } x \in \mathcal{C}^{k+n}(I, \mathbb{C}).$$

We then define the non-differentiable embedding of equation (13) as follow:

Definition 11. — The non-differentiable embedding of equation (13) is defined by

$$(14) \quad \text{Emb}_{\square} (O_{\mathbf{f}}^{\mathbf{g}}) \left(x, \frac{\square x}{\square t}, \dots, \frac{\square^k x}{\square t^k} \right) = 0, \quad x \in \mathcal{C}_{\square}^{k+n}(I, \mathbb{C}^d),$$

where $\mathcal{C}_{\square}^{k+n}(I, \mathbb{C}^d)$ denotes the set of functions $x \in \mathcal{C}^0(I, \mathbb{C}^d)$ such that $\square^i x \in \mathcal{C}^0(I, \mathbb{C}^d)$ for $i = 1, \dots, k+n$.

Note that as long as the form of the operator is fixed the non-differentiable embedding procedure associates a *unique* equation.

In the following part, we explicit the non-differentiable embedding of a particular class of differential equations called Lagrangian systems.

PART II

NON-DIFFERENTIABLE EMBEDDING OF LAGRANGIAN SYSTEMS
AND COHERENCE PRINCIPLE

1. Non-differentiable embedding for Euler-Lagrange equation

1.1. Reminder about Lagrangian systems. — Lagrangian systems play a central role in dynamical systems and physics, in particular for classical mechanics. We refer to [2] for more details.

Definition 12. — An admissible Lagrangian function L is a function $L : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d \rightarrow \mathbb{C}$ such that $L(t, x, v)$ is holomorphic with respect to v , differentiable with respect to x and real when $v \in \mathbb{R}$.

A Lagrangian function defines a functional on $\mathcal{C}^1(I, \mathbb{R})$, denoted by

$$(15) \quad \mathcal{L}_{a,b} : \mathcal{C}^1(I, \mathbb{R}^d) \rightarrow \mathbb{R}, \quad x \in \mathcal{C}^1(I, \mathbb{R}^d) \mapsto \int_a^b L(s, x(s), \frac{dx}{dt}(s)) ds.$$

When no confusion is possible we will simply write \mathcal{L} for $\mathcal{L}_{a,b}$.

The classical *calculus of variations* analyzes the behavior of \mathcal{L} under small perturbations of the initial function x . The main ingredient is a notion of differentiable functional and extremal.

Definition 13 (Space of variations). — We denote by $V(a, b)$ the set of functions h in $\mathcal{C}^1(I, \mathbb{R}^d)$ such that $h(a) = h(b) = 0$.

A functional \mathcal{L} is $V(a, b)$ -differentiable at point $\gamma \in \mathcal{C}^1(I, \mathbb{R}^d)$ if and only if

$$\mathcal{L}(\gamma + \theta h) - \mathcal{L}(\gamma) = \theta D\mathcal{L}(\gamma)(h) + o(\theta),$$

for $\theta > 0$ sufficiently small and any $h \in V(a, b)$.

Definition 14. — An extremal for the functional \mathcal{L} is a function $\gamma \in \mathcal{C}^1(I, \mathbb{R}^d)$ such that $D\mathcal{L}(\gamma)(h) = 0$ for any $h \in V(a, b)$, where $D\mathcal{L}(\gamma)(h)$ is the Fréchet derivative of \mathcal{L} at point γ in the direction h .

Extremals of the functional \mathcal{L} can be characterized by an ordinary differential equation of order 2, called the Euler-Lagrange equation.

Theorem 2. — The extremals of \mathcal{L} coincide with the solutions of the Euler-Lagrange equation denoted by (EL) and defined by

$$\frac{d}{dt} \left[\frac{\partial L}{\partial v} \left(t, \gamma(t), \frac{d\gamma}{dt}(t) \right) \right] = \frac{\partial L}{\partial x} \left(t, \gamma(t), \frac{d\gamma}{dt}(t) \right). \quad (EL)$$

This equation can be seen as the action of the differential operator

$$(16) \quad O_{(EL)} = \frac{d}{dt} \circ \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x}$$

on $(t, \gamma(t), \frac{d\gamma}{dt}(t))$. The Euler-Lagrange equation (EL) is then

$$O_{(EL)}((t, \gamma(t), \frac{d\gamma}{dt}(t))) = 0.$$

1.2. Non-differentiable Euler-Lagrange equation. — The non-differentiable embedding procedure allows us to define a natural extension of the classical Euler-Lagrange equation in the non-differentiable context.

Lemma 3. — *Let L be an admissible Lagrangian function. The non-differentiable embedding of the Euler-Lagrange differential operator $O_{(EL)}$ is given by*

$$(17) \quad \text{Emb}_{\square}(O_{(EL)}) = \frac{\square}{\square t} \circ \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x}.$$

Proof. — The operator (16) is first considered as acting on $I \times \mathcal{C}^1(I, \mathbb{R}^d) \times \mathcal{C}^0(I, \mathbb{C}^d)$, i.e. for all $(t, x(t), y(t)) \in I \times \mathcal{C}^1(I, \mathbb{R}^d) \times \mathcal{C}^0(I, \mathbb{C}^d)$ we have

$$O_{(EL)}(t, x(t), y(t)) = \frac{d}{dt} \left(\frac{\partial L}{\partial v}(t, x(t), y(t)) \right) - \frac{\partial L}{\partial x}(t, x(t), y(t)).$$

This operator is of the form $O_{\mathbf{f}}^{\mathbf{g}}$ with

$$\mathbf{f} = \left(\mathbf{1}, \frac{\partial L}{\partial x} \right),$$

and

$$\mathbf{g} = \left(-\frac{\partial L}{\partial v}, \mathbf{1} \right),$$

where $\mathbf{1} : \mathbb{R} \times \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}$ is the *constant* function $\mathbf{1}(t, x, y) = 1$. As a consequence, $O_{(EL)}$ is given by

$$O_{(EL)} = \mathbf{1} \cdot \frac{d}{dt} \circ \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} \cdot \text{Id} \circ \mathbf{1},$$

with the convention that $\left(\frac{d}{dt} \right)^0 = \text{Id}$. We then obtain equation (17) using definition 9. \square

Remark 3. — *Another possible choice would be $\mathbf{f} = (\mathbf{1}, \mathbf{1})$ and $\mathbf{g} = \left(-\frac{\partial L}{\partial v}, \frac{\partial L}{\partial x} \right)$, i.e. exchanging the f_0 and g_0 terms. This alternative is always possible, but we use the convention to put the dependence on f_0 because the usual form of a differential operator is $\sum_i a_i \frac{d^i}{dt^i}$ which can be written only with a component \mathbf{f} . Moreover, taking this alternative for the 0-term in \mathbf{f} and \mathbf{g} does not affect the resulting form of the embed equation (this is of course not the case when dealing with i -th terms, $i \geq 1$).*

Theorem 3. — *Let L be an admissible Lagrangian function. The non-differentiable embedded Euler-Lagrange equation, denoted by $\text{Emb}(EL)$ associated to L is given by*

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial v}(t, \gamma(t), \frac{\square}{\square t} \gamma(t)) \right) - \frac{\partial L}{\partial x}(t, \gamma(t), \frac{\square}{\square t} \gamma(t)) = 0. \quad \text{Emb}(EL)$$

Proof. — Using definition 11, the non-differentiable embedding of equation (EL) is given by

$$\text{Emb}_{\square} (O_{(EL)}) \left(x, \frac{\square}{\square t} x \right) = 0,$$

which reduces to equation $\text{Emb}(EL)$ thanks to lemma 3. \square

2. Embedding of Lagrangian systems

In this section, we derive the non-differentiable embedding of a particular class of ordinary differential equations called *Euler-Lagrange equations* which governs the dynamics of *Lagrangian systems*.

2.1. Embedding of the Lagrangian functional. —

Definition 15. — *Let be given a Lagrangian functional $\mathcal{L}_{a,b}$ as defined in (15). The natural embedding of the Lagrangian functional $\mathcal{L}_{a,b}$ is given by*

$$(18) \quad \mathcal{L}_{\square} : \mathcal{C}_{\square}^1(I, \mathbb{R}^d) \rightarrow \mathbb{R}, \quad x \in \mathcal{C}_{\square}^1(I, \mathbb{R}^d) \mapsto \int_a^b L(s, x(s), \frac{\square}{\square t} x(s)) ds.$$

2.2. Non-differentiable calculus of variations. — Let α be a real number $0 < \alpha < 1$ and ϵ be a parameter which is assumed to be sufficiently small, *i.e.* $0 < \epsilon \ll 1$, without specifying its exact smallness.

Definition 16. — *Let $\gamma \in H^{\alpha}(I, \mathbb{R}^d)$. Let V be a subvectorial space of $H_0^{\beta} := \{h \in H^{\beta}(I, \mathbb{R}^d), h(a) = h(b) = 0\}$, with $\alpha + \beta > 1$, the space of non-differentiable variations. A V -variation γ' of γ is a curve in $\mathcal{C}^0(I, \mathbb{R}^d)$ defined by*

$$\gamma'(t) := \gamma(t) + h(t), \quad h \in V.$$

Such a curve is denoted by $\gamma' := \gamma + h$.

Definition 17. — *Let $\Phi : \mathcal{C}^0(I, \mathbb{R}^d) \rightarrow \mathbb{C}$ be a functional. The functional Φ is called V -differentiable on a curve $\gamma \in H^{\alpha}(I, \mathbb{R}^d)$ if and only if its Fréchet differential*

$$\lim_{\epsilon \rightarrow 0} \frac{\Phi(\gamma + \epsilon h) - \Phi(\gamma)}{\epsilon}$$

exists in any direction $h \in V$. And then $D\Phi$ is called its differential and is given by

$$D\Phi(\gamma)(h) = \lim_{\epsilon \rightarrow 0} \frac{\Phi(\gamma + \epsilon h) - \Phi(\gamma)}{\epsilon}.$$

Definition 18. — (*V-extremal curves*) A *V-extremal curve* of the functional Φ on the space V of curves of $H^\alpha(I, \mathbb{R}^d)$ is a curve $\gamma \in H^\alpha(I, \mathbb{R}^d)$ satisfying

$$D\Phi(\gamma)(h) = 0,$$

for any $h \in V$.

The following theorem gives the analogous of the Euler-Lagrange equations for extremals of our functionals.

Theorem 4. — The differential of \mathcal{L}_\square on $\gamma \in H^\alpha(I, \mathbb{R}^d) \cap \mathcal{C}_\square^1(I, \mathbb{R}^d)$ is given by

$$(19) \quad D\mathcal{L}_\square(\gamma)(h) = \int_a^b \left(\frac{\partial L}{\partial x}(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) \cdot h(t) + \frac{\partial L}{\partial v}(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) \cdot \frac{\square h(t)}{\square t} \right) dt.$$

for any $h \in V$.

Proof. — Let $\Phi : H^\alpha(I, \mathbb{R}^d) \rightarrow \mathbb{C}$ be the functional defined by

$$(20) \quad \Phi(\gamma) = \int_a^b L(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) dt,$$

for any $\gamma \in H^\alpha(I, \mathbb{R}^d)$. With help of a Taylor expansion, we obtain its differential given by :

$$D\Phi(\gamma)(h) = \int_a^b \left(\frac{\partial L}{\partial x}(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) h(t) + \frac{\partial L}{\partial v}(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) \frac{\square h(t)}{\square t} \right) dt.$$

□

Theorem 5 (Non-differentiable least-action principle). — Let $0 < \alpha < 1$, $\alpha + \beta > 1$. 1. Let L be an admissible Lagrangian function of class \mathcal{C}^2 . We assume that $\gamma \in H^\alpha(I, \mathbb{R}^d)$, and $\frac{\square\gamma}{\square t} \in H^\alpha(I, \mathbb{R}^d)$. A curve $\gamma \in H^\alpha(I, \mathbb{R}^d)$ satisfying the following generalized Euler-Lagrange equation

$$\frac{\partial L}{\partial x}(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) - \frac{\square}{\square t} \left(\frac{\partial L}{\partial v}(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) \right) = 0. \quad (NDEL)$$

is an extremal curve of the functional (18) on the space of variations $V = H_0^\beta$.

Proof. — As $\frac{\partial L}{\partial v}(t, \gamma(t), \frac{\square\gamma(t)}{\square t})$ is $H^\alpha(I, \mathbb{R}^d)$, and $h \in V = H_0^\beta(I, \mathbb{R}^d)$, with $\alpha + \beta > 1$, we obtain using lemma 1

$$\int_a^b \frac{\partial L}{\partial v} \cdot \frac{\square h(t)}{\square t} dt = \int_a^b \frac{\square}{\square t} \left(\frac{\partial L}{\partial v} \cdot h \right) dt - \int_a^b \frac{\square}{\square t} \left(\frac{\partial L}{\partial v} \right) \cdot h.$$

Using lemma 2 and the fact that $h(a) = h(b) = 0$, we obtain that

$$\int_a^b \frac{\square}{\square t} \left(\frac{\partial L}{\partial v} \cdot h \right) dt = 0.$$

The differential (19) becomes

$$(21) \quad D\mathcal{L}_\square(\gamma)(h) = \int_a^b \left[\frac{\partial L}{\partial x}(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) - \frac{\square}{\square t} \left(\frac{\partial L}{\partial v}(t, \gamma(t), \frac{\square\gamma(t)}{\square t}) \right) \right] \cdot h(t) dt,$$

for all $h \in V = H_0^\beta$. □

3. Coherence problem

We prove that the non-differentiable embedding is coherent, i.e. that the embedded Euler-Lagrange equation coincides with the non-differentiable Euler-Lagrange equation obtained using the non-differentiable calculus of variations. We discuss the compatibility between the non-differentiable embedding of Lagrangian systems and the non-differentiable calculus of variations using the notion of coherent embedding.

Definition 19. — *An embedding procedure is called coherent when the two Euler-Lagrange equations are the same, i.e. if*

$$\text{NDEL} = \text{Emb}(EL),$$

assuming that NDEL is obtained from the embedding of the classical functional using the same embedding procedure.

We have the following result:

Theorem 6. — *(Coherence principle) Let L be an admissible Lagrangian function, then the following diagram commutes*

$$(22) \quad \begin{array}{ccc} \mathcal{L}(x) & \xrightarrow{\text{Emb}_\square} & \mathcal{L}_\square(x) \\ \begin{array}{c} \downarrow \text{L.A.P.} \\ \frac{d}{dt} \left(\frac{\partial L}{\partial v}(z(t)) \right) = \frac{\partial L}{\partial x}(z(t)) \end{array} & \xrightarrow{\text{Emb}_\square} & \begin{array}{c} \downarrow \text{N.D.L.A.P.} \\ \frac{\square}{\square t} \left(\frac{\partial L}{\partial v}(Z(t)) \right) = \frac{\partial L}{\partial x}(Z(t)) \end{array} \end{array}$$

where $z(t) = (x(t), \dot{x}(t))$, $Z(t) = (x(t), \frac{\square x(t)}{\square t})$, and LAP stands for "Least-Action Principle" and ND LAP for the Non-Differentiable Least-Action Principle.

Remark 4. — *An embedding procedure is not always coherent. We refer to [9] and [7] where the coherence problem is discussed for the stochastic and the fractional embedding respectively.*

This result gives a strong support to the non-differentiable embedding of Lagrangian equations. Indeed, we proved that the non-differentiable Euler-Lagrange equation is derived from an extended variational principle constructed on the non-differentiable embedding of the classical functional.

In the following part, we prove that the solutions of Schrödinger equation as well as the ones of Navier-Stokes equation can be seen as solutions of a non-differentiable Euler-Lagrange equation.

PART III
NON-DIFFERENTIABLE HAMILTONIAN SYSTEMS

4. Reminder about Hamiltonian systems

Let L be an admissible Lagrangian function. If L satisfies the so-called Legendre property, we can associate to L , a Hamiltonian function denoted by H . Indeed, the basic idea underlying the Hamiltonian formalism is to code the dichotomy between speed and position.

Definition 20. — *Let L be an admissible Lagrangian function. The Lagrangian L is said to satisfy the Legendre property if the mapping $v \mapsto \frac{\partial L}{\partial v}(t, x, v)$ is invertible for any $(t, x, v) \in I \times \mathbb{R}^d \times \mathbb{C}^d$.*

As a consequence, if we introduce a new variable

$$p = \frac{\partial L}{\partial v}(t, x, v)$$

and L satisfies the Legendre property we can find a function f such that

$$v = f(t, x, p).$$

Using this notation, we have the following definition.

Definition 21. — *Let L be an admissible Lagrangian function satisfying the Legendre property. The Hamiltonian function H associated to L , is given by*

$$\begin{aligned} H : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d &\longrightarrow \mathbb{C} \\ (t, x, p) &\longmapsto H(t, x, p) = pf(t, x, p) - L(t, x, f(t, x, p)). \end{aligned}$$

A natural Lagrangian function is usually of the form

$$L(t, x, v) = \frac{1}{2}mv^2 - U(x),$$

where $m > 0$. The term $\frac{1}{2}mv^2$ is the kinetic energy and $U(x)$ the potential energy. Then the Hamiltonian function associated to L is

$$H(t, x, p) = \frac{1}{2m}p^2 + U(x)$$

and represents the total energy of the system.

The dynamics generated by a Hamiltonian system is defined as follow:

Proposition 1. — *(Hamilton's least-action principle, [2]) A curve $(t, x(t), p(t)) \in I \times \mathbb{R}^d \times \mathbb{C}^d$ is an extremal of the Hamiltonian functional*

$$(23) \quad \mathcal{H}(x, p) = \int_a^b p(t) \frac{dx}{dt}(t) - H(t, x(t), p(t)) dt$$

if and only if it satisfies the Hamiltonian system associated to H given by

$$(24) \quad \begin{cases} \frac{dx}{dt}(t) &= \frac{\partial H}{\partial p}(t, x(t), p(t)), \\ \frac{dp}{dt}(t) &= -\frac{\partial H}{\partial x}(t, x(t), p(t)). \end{cases}$$

5. The non-differentiable case

The non-differentiable embedding induces a change in the phase space with respect to the classical case. As a consequence, we have to work with variables (x, p) which belongs to $\mathbb{R}^d \times \mathbb{C}^d$ and not only to $\mathbb{R}^d \times \mathbb{R}^d$ as usual. This means that we must embed the Hamiltonian system in $\mathcal{C}^0(I, \mathbb{R}^d) \times \mathcal{C}^0(I, \mathbb{C}^d)$

Lemma 4. — 1. The embedded Hamiltonian system (24) is given by:

$$(25) \quad \begin{cases} \frac{\square x}{\square t}(t) &= \frac{\partial H}{\partial p}(t, x(t), p(t)), \\ \frac{\square p}{\square t}(t) &= -\frac{\partial H}{\partial x}(t, x(t), p(t)). \end{cases}$$

2. The embedded Hamiltonian functional \mathcal{H}_{\square} is defined on $H^{\alpha}(I, \mathbb{R}^d) \times H^{\alpha}(I, \mathbb{C}^d)$ by:

$$\mathcal{H}_{\square}(x, p) = \int_a^b p(t) \frac{\square x(t)}{\square t} - H(t, x(t), p(t)) dt.$$

Using the non-differentiable calculus of variations we can derive the Euler-Lagrange equation for \mathcal{H}_{\square} .

Theorem 7 (Non-differentiable Hamilton's least-action principle)

Let L be an admissible Lagrangian function satisfying the Legendre property, and H the corresponding Hamiltonian defined by (23). We suppose H is of class \mathcal{C}^2 , and $(x, p) \in H^{\alpha}(I, \mathbb{R}^d) \times \mathcal{C}^0(I, \mathbb{C}^d)$, with $\frac{1}{2} \leq \alpha < 1$.

The curve $\gamma : t \rightarrow (t, x(t), p(t)) \in I \times \mathbb{R}^d \times \mathbb{C}^d$ is an extremal of the functional

$$\mathcal{H}_{\square}(x, p) = \int_a^b p(t) \frac{\square x(t)}{\square t} - H(t, x(t), p(t)) dt,$$

if and only if it satisfies the non-differentiable Hamiltonian system (25).

Proof. — The functional \mathcal{H}_{\square} is a Lagrangian functional with Lagrangian \mathbb{L} given by

$$\mathbb{L} : \begin{array}{ccc} \mathbb{R}^d \times \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d & \longrightarrow & \mathbb{C}, \\ (x, p, v, w) & \longmapsto & \mathbb{L}(x, p, v, w) = pv - H(x, p), \end{array}$$

the Lagrangian being evaluated on $(x(t), p(t), \frac{\square x}{\square t}, \frac{\square p}{\square t})$. As $(x, p) \in H^\alpha(I, \mathbb{R}^d) \times H^\alpha(I, \mathbb{C}^d)$, $\frac{1}{2} \leq \alpha < 1$, the non-differentiable Euler-Lagrange equation is given by

$$\begin{cases} \frac{\square}{\square t} \left(\frac{\partial \mathbb{L}}{\partial v} \left(x, p, \frac{\square x}{\square t}, \frac{\square p}{\square t} \right) \right) = \frac{\partial \mathbb{L}}{\partial x} \left(x, p, \frac{\square x}{\square t}, \frac{\square p}{\square t} \right), \\ \frac{\square}{\square t} \left(\frac{\partial \mathbb{L}}{\partial w} \left(x, p, \frac{\square x}{\square t}, \frac{\square p}{\square t} \right) \right) = \frac{\partial \mathbb{L}}{\partial p} \left(x, p, \frac{\square x}{\square t}, \frac{\square p}{\square t} \right), \end{cases}$$

which leads to

$$\begin{cases} \frac{\square p}{\square t} = -\frac{\partial H}{\partial x}, \\ 0 = \frac{\square x}{\square t} - \frac{\partial H}{\partial p}. \end{cases}$$

This concludes the proof. \square

Again we have coherence of our embedding procedure with respect to the Hamiltonian formalism.

Corollary 2. — (Coherence) *Let H be a Hamiltonian function, then the following diagram commutes*

$$(26) \quad \begin{array}{ccc} \mathcal{H}(x, p) & \xrightarrow{\text{Emb}_\square} & \mathcal{H}_\square(x, p) \\ \begin{array}{c} \text{L.A.P.} \\ \downarrow \end{array} & & \begin{array}{c} \downarrow \\ \text{N.D.L.A.P.} \end{array} \\ \left\{ \begin{array}{l} \frac{dx}{dt} = \frac{\partial H}{\partial p}, \\ \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \end{array} \right. & \xrightarrow{\text{Emb}_\square} & \left\{ \begin{array}{l} \frac{\square x}{\square t} = \frac{\partial H}{\partial p}, \\ \frac{\square p}{\square t} = -\frac{\partial H}{\partial x}. \end{array} \right. \end{array}$$

PART IV

APPLICATION TO PDE'S

1. Non-differentiable method of characteristics

The classical method of characteristics for a PDE is to look for curves $s \mapsto (x(s), t(s))$ where $x(s)$ and $t(s)$ are solutions of an ordinary differential equation such that solutions $u(x, t)$ of the PDE satisfies

$$\frac{d}{ds} (u(x(s), t(s))) = F(x(s), t(s)),$$

where F is the non homogeneous part of the PDE.

In many cases, we can choose

$$\frac{dt}{ds} = 1$$

so that one is reduced to find a curve $t \rightarrow x(t)$ satisfying the following ordinary differential equation

$$\frac{d}{dt} (u(x(t), t)) = F(x(t), t).$$

The method of characteristics does not work for parabolic PDEs and PDEs of mixed type like hyperbolic/parabolic (as for example the transport equation with diffusion).

Using the operator $\frac{\square}{\square t}$ one can generalize this method. We say that a curve $s \rightarrow (x(s), t(s))$ is a non-differentiable characteristic for a given PDE if the solution $u(x, t)$ satisfies

$$\frac{\square}{\square s} (u(x(s), t(s))) = F(x(s), t(s)),$$

and x and t satisfy an ordinary differential equation in $\frac{\square}{\square t}$.

In the following section, we characterize the non-differentiable characteristics of the Navier-Stokes equations.

2. Non-differentiable characteristics for the Navier-Stokes equation

There exist already known tentative for deriving the equations of fluids mechanics from a variational principle. We refer in particular to the work of Arnold [1] and ([10],[3]) which deal with an interpretation of the Euler equation as geodesics on an infinite Lie group. For an overview of results and questions in fluid mechanics we refer to ([11],[17]).

In this section, we characterize non-differentiable characteristics for the Navier-Stokes equation as critical point of a non-differentiable Lagrangian functional.

The incompressible homogeneous Navier-Stokes equation looks like

$$(27) \quad \begin{cases} \frac{\partial u}{\partial t} + \sum_{k=1}^d u_k \frac{\partial u}{\partial x_k} = \nu \Delta_x u - \nabla_x p, \\ \operatorname{div} u = 0, \end{cases}$$

where the unknown are the velocity $u(t, x) \in \mathbb{R}^d$, $u = (u_1, \dots, u_d)$, and the pressure $p(t, x) \in \mathbb{R}$. The constant $\nu \in \mathbb{R}^+$ is the viscosity. Equation (27) is also equivalent to

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{k=1}^d u_k \frac{\partial u_i}{\partial x_k} = \nu \Delta_x u_i - \frac{\partial p}{\partial x_i} & i = 1, \dots, d, \\ \operatorname{div} u = 0. \end{cases}$$

We look for a curve $t \rightarrow x(t)$ such that

$$\frac{\square}{\square t} (u(x(t), t)) = -\nabla_x p.$$

The solution u of equation (27) is such that $u \in \mathcal{C}^2(I \times \mathbb{R}^d, \mathbb{R})$, then if we define $x := u + W$, $x \in \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$ with $\frac{1}{2} \leq \alpha < 1$ then from theorem 1, we have for any $i = 1, \dots, d$

$$\frac{\square u_i(x(t), t)}{\square t} = \nabla_x u_i(x(t), t) \cdot \nabla_{\square} x(t) + \frac{\partial u_i}{\partial t}(x(t), t) + \sum_{k=1}^d \sum_{l=1}^d \frac{1}{2} \frac{\partial^2 u_i}{\partial x_k \partial x_l}(x(t), t) a_{k,l}(x(t)).$$

We consider $x = (x_1, \dots, x_d) \in \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$ of the form

$$(28) \quad x_i(t) = \int_0^t u_i(x(s), s) ds + W_i(t), \quad W_i \in \mathbb{H}^\alpha, \frac{1}{2} \leq \alpha < 1, \quad i = 1, \dots, d.$$

Then, $\nabla_{\square} x(t) = u(x(t), t)$. As a consequence, we obtain for any $i = 1, \dots, d$

$$\frac{\square u_i(x(t), t)}{\square t} = \nabla_x u_i(x(t), t) \cdot u(x(t), t) + \frac{\partial u_i}{\partial t}(x(t), t) + \sum_{k=1}^d \sum_{l=1}^d \frac{1}{2} \frac{\partial^2 u_i}{\partial x_k \partial x_l}(x(t), t) a_{k,l}(W_i(t)).$$

We choose $W = (W_1, \dots, W_d)$ such that

$$(29) \quad a_{k,l}(W(t)) = \begin{cases} -2\nu & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

Note that since u satisfies the Navier-stokes equation, $\frac{\square}{\square t} (u_i(x(t), t)) = -\frac{\partial p}{\partial x_i}$. We do not have any explicit expression of W , nevertheless, in the case of stochastic process, we know it is possible to construct W , [8]. We then introduce the following space:

Definition 22. — We denote by $\mathcal{C}_{\text{nav}}^{1,\alpha}$, $\frac{1}{2} \leq \alpha < 1$, the subset of $\mathcal{C}^{1,\alpha}$ defined by:

$$\begin{aligned} \mathcal{C}_{\text{nav}}^{1,\alpha} &:= \{x = (x_1, \dots, x_d) \in \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d), x_i(t) = \int_0^t u_i(x(s), s) ds + W_i(t), \\ &\quad W_i \in \mathbb{H}^\alpha, \frac{1}{2} \leq \alpha < 1, i = 1, \dots, d\}, \end{aligned}$$

where u is a solution of the Navier-Stokes equation and $W = (W_1, \dots, W_d)$ satisfies (3).

Let note that on $\mathcal{C}_{\text{nav}}^{1,\alpha}$ the non-differentiable characteristics satisfy by definition

$$(30) \quad \frac{\square}{\square t} (u(x(t), t)) = -\nabla_x p,$$

which can be rewritten as

$$(31) \quad \frac{\square}{\square t} \left(\frac{\square x}{\square t} \right) = -\nabla_x p.$$

This equation looks like a non-differentiable Euler-Lagrange equation. In fact, one can prove that non-differentiable characteristics of the Navier-Stokes equation correspond to critical point of a non-differentiable Lagrangian functional.

Theorem 8. — *The non-differentiable characteristics $x \in \mathcal{C}_{\text{nav}}^{1,\alpha}$, $\frac{1}{2} \leq \alpha < 1$ of the Navier-Stokes equations correspond to $\mathcal{C}_{\text{nav}}^{1,\alpha}$ extremals of the Lagrangian*

$$(32) \quad L(t, x, v) = \frac{1}{2}v^2 - p(x, t),$$

where p is the pressure.

Proof. — Theorem 5 adapted to functions in $\mathcal{C}_{\text{nav}}^{1,\alpha}$, gives the non-differentiable extremals of (32) over $\mathcal{C}_{\text{nav}}^{1,\alpha}$ by

$$\frac{\square}{\square t} \left(\frac{\partial L}{\partial v}(t, x, \frac{\square x}{\square t}) \right) = \frac{\partial L}{\partial x}(t, x, \frac{\square x}{\square t}), \quad x \in \mathcal{C}_{\text{nav}}^{1,\alpha}.$$

As $\frac{\partial L}{\partial v} = v$ and $\frac{\partial L}{\partial x} = -\nabla_x p$, we obtain (31) which coincides with (30) over $\mathcal{C}_{\text{nav}}^{1,\alpha}$. \square

We have now a clear understanding of the different roles of the pressure and the viscosity terms:

- the pressure $p(x)$ plays the role of a potential for the underlying classical dynamics,
- the viscosity ν controls the irregularity of solutions via W .

It is not easy to characterize the set $\mathcal{C}_{\text{nav}}^{1,\alpha}$ and more work are need in this direction. We only remark that this condition, on the quadratic part of the velocity can be realized in the stochastic setting using diffusion processes with constant diffusion.

3. The Schrödinger equation

In [5], we already proved that we can recover the Schrödinger equation in dimension one using an ϵ -dependent embedding (See Remark 2) and a non-differentiable variational principle. This is not satisfying due to the dependence in ϵ . Using the new framework we defined in part II, we prove that the solutions of the Schrödinger equation in \mathbb{R}^d , $d \geq 1$, can be seen as extremals of the non-differentiable embedded Newton Lagrangian in $\mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$, $\frac{1}{2} \leq \alpha < 1$ under specific geometrical constraints.

In the following, we denote by ψ an application

$$\psi : \begin{array}{ccc} \mathbb{R}^d \times \mathbb{R} & \longrightarrow & \mathbb{C} \\ (x, t) & \longmapsto & \psi(x, t). \end{array}$$

Definition 23. — We denote by $\mathcal{C}_{\text{schr}}^\alpha(\gamma, \delta) \subset \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$ the set of $x \in \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$ such that

$$(33) \quad \begin{cases} \frac{\square}{\square t} x_j(t) &= -2i\gamma \frac{\partial \ln \psi}{\partial x_j}(x(t), t), \quad j = 1, \dots, d, \\ a_{k,k}(x(t)) &= \delta, \quad k = 1, \dots, d, \\ a_{k,l}(x(t)) &= 0 \text{ if } k, l = 1, \dots, d, \quad k \neq l \end{cases}$$

where $\gamma \in \mathbb{R}$ and $\delta \in \mathbb{C}$, and $\psi : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ is $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}, \mathbb{C})$.

Let h be the Planck constant, m the mass of a particle, $\bar{h} = h/2\pi$ the reduced Planck constant. The dynamic of a particle of mass m in \mathbb{R}^d , $d \geq 1$, under the potential $U : \mathbb{R}^d \rightarrow \mathbb{R}$ in Quantum mechanics is governed by the *Schrödinger equation* :

$$(34) \quad i\bar{h} \frac{\partial \psi}{\partial t} + \frac{\bar{h}^2}{2m} \sum_{k=1}^d \frac{\partial^2 \psi}{\partial x_k^2} = U(x)\psi,$$

where $\psi : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ is the *wave function* associated to the particle.

Theorem 9. — The solutions of the Schrödinger equation (34) coincide on $\mathcal{C}_{\text{schr}}^\alpha(\bar{h}/2m, -i\bar{h}/m)$ with $\frac{1}{2} \leq \alpha < 1$ extremals of the classical Newton's Lagrangian

$$L(t, x, v) = \frac{1}{2}mv^2 - U(x).$$

Proof. — By theorem 5, extremals of our functional satisfy the non-differentiable Euler-Lagrange equation

$$(35) \quad \frac{\square}{\square t} \left(m \frac{\square x_j(t)}{\square t} \right) = -\frac{\partial U}{\partial x_j}(x), \quad j = 1, \dots, d.$$

Since $x \in \mathcal{C}_{\text{schr}}^\alpha(\gamma, \delta)$, x satisfy (3):

$$(36) \quad \frac{\square}{\square t} \left[\frac{\square x_j}{\square t} \right] = -i2\gamma \frac{\square}{\square t} \left(\frac{\partial \ln(\psi)}{\partial x_j}(x(t), t) \right).$$

By theorem 1 with $n = 2$, we have

$$(37) \quad \begin{aligned} \frac{\square}{\square t} \left[\frac{\square x_j}{\square t} \right] &= -i2\gamma \sum_{k=1}^d \frac{\square x_k}{\square t} \frac{\partial}{\partial x_k} \left(\frac{\partial \ln \psi}{\partial x_j} \right) (x(t), t) - i2\gamma \frac{\partial}{\partial t} \left(\frac{\partial \ln(\psi)}{\partial x_j} \right) (x(t), t) \\ &\quad - i2\gamma \sum_{k=1}^d \sum_{l=1}^d \frac{1}{2} a_{k,l}(x(t)) \frac{\partial^2}{\partial x_k \partial x_l} \left(\frac{\partial \ln \psi}{\partial x_j} \right) (x(t), t). \end{aligned}$$

Replacing $\frac{\square x_k}{\square t}$ by its expression (from (3)) as a function of ψ , we obtain

$$(38) \quad \begin{aligned} \frac{\square x_k}{\square t} \frac{\partial}{\partial x_k} \left(\frac{\partial \ln \psi}{\partial x_j} \right) (x(t), t) &= -i2\gamma \frac{\partial \ln \psi}{\partial x_k} \frac{\partial}{\partial x_k} \left(\frac{\partial \ln \psi}{\partial x_j} \right) (x(t), t), \\ &= -i\gamma \frac{\partial}{\partial x_j} \left[\frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial x_k} \right)^2 \right] (x(t), t). \end{aligned}$$

Combining equations (37) and (38), and using the definition of $a_{k,l}$ from (3), we get

$$\begin{aligned} \frac{\square}{\square t} \left[\frac{\square x_j}{\square t}(x(t), t) \right] &= -i2\gamma \frac{\partial}{\partial x_j} \left[-i\gamma \frac{1}{\psi^2} \sum_{k=1}^d \left(\frac{\partial \psi}{\partial x_k} \right)^2 + \frac{1}{\psi} \frac{\partial \psi}{\partial t} + \frac{\delta}{2} \sum_{k=1}^d \left(\frac{1}{\psi} \frac{\partial^2 \psi}{\partial x_k^2} - \frac{1}{\psi^2} \left(\frac{\partial \psi}{\partial x_k} \right)^2 \right) \right] (x(t), t) \\ &= -i2\gamma \frac{\partial}{\partial x_j} \left[- \left(i\gamma + \frac{\delta}{2} \right) \frac{1}{\psi^2} \sum_{k=1}^d \left(\frac{\partial \psi}{\partial x_k} \right)^2 + \frac{1}{\psi} \frac{\partial \psi}{\partial t} + \frac{\delta}{2} \frac{1}{\psi} \sum_{k=1}^d \frac{\partial^2 \psi}{\partial x_k^2} \right] (x(t), t). \end{aligned}$$

The non-differentiable Euler-Lagrange equation (35) becomes

$$-i2\gamma m \frac{\partial}{\partial x_j} \left[- \left(i\gamma + \frac{\delta}{2} \right) \frac{1}{\psi^2} \sum_{k=1}^d \left(\frac{\partial \psi}{\partial x_k} \right)^2 + \frac{1}{\psi} \frac{\partial \psi}{\partial t} + \frac{\delta}{2} \frac{1}{\psi} \sum_{k=1}^d \frac{\partial^2 \psi}{\partial x_k^2} \right] = - \frac{\partial U}{\partial x_j}.$$

In order to recover the Schrödinger equation, one must choose δ and γ such that

$$-2i\gamma m = -i\bar{h}, \quad -i\gamma m \delta = -\bar{h}^2/2m, \quad \text{and} \quad i\gamma + \frac{\delta}{2} = 0.$$

This system is overdetermined. The two first equations lead to

$$\gamma = \frac{\bar{h}}{2m}, \quad \delta = -i \frac{\bar{h}}{m}.$$

We verify that the third constraint is satisfied.

As a consequence, for each $j = 1, \dots, d$, we obtain

$$\frac{\partial}{\partial x_j} \left[- \frac{i\bar{h}}{\psi} \frac{\partial \psi}{\partial t} - \frac{\bar{h}^2}{2m} \frac{1}{\psi} \sum_{k=1}^d \frac{\partial^2 \psi}{\partial x_k^2} + U \right] (x(t), t) = 0,$$

meaning that the following equality is true

$$i\bar{h} \frac{\partial \psi}{\partial t} + \frac{\bar{h}^2}{2m} \sum_{k=1}^d \frac{\partial^2 \psi}{\partial x_k^2} = U(x)\psi + c(x)$$

where $c(x)$ is an arbitrary function. This concludes the proof. \square

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NOTATIONS

- $d \in \mathbb{N}$ a fixed integer.
- I an open set in \mathbb{R} .
- $a, b \in \mathbb{R}$, $a < b$, such that $[a, b] \subset I$.
- $\mathcal{F}(I, \mathbb{R}^d)$ the set of functions from I to \mathbb{R}^d .
- $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) the set of continuous functions.
- $\mathcal{C}_{\square}^1(I, \mathbb{R})$ the set of continuous functions $f \in \mathcal{C}^0(I, \mathbb{R})$ such that $\frac{\square f}{\square t} \in \mathcal{C}^0(I, \mathbb{R})$.
- $\mathcal{C}^n(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^n(I, \mathbb{C}^d)$) the set of functions in $\mathcal{C}^0(I, \mathbb{R}^d)$ (respectively $\mathcal{C}^0(I, \mathbb{C}^d)$) which are differentiable up to order n .
- $\mathcal{C}^{k,\alpha}(I, \mathbb{C}^d)$ the subset of $\mathcal{C}^0(I, \mathbb{C}^d)$ of functions $x : I \rightarrow \mathbb{C}^d$ of the form $x := u + w$, where $u \in \mathcal{C}^k(I, \mathbb{C}^d)$ and $w \in \mathbb{H}^\alpha(I, \mathbb{C}^d)$.
- $\mathcal{C}_{\square}^1(I, \mathbb{R})$ the set of continuous functions $f \in \mathcal{C}^0(I, \mathbb{R})$ such that $\frac{\square f}{\square t} \in \mathcal{C}^0(I, \mathbb{R})$.
- $\mathcal{C}_{\text{nav}}^{1,\alpha}$, $\frac{1}{2} \leq \alpha < 1$, the subset of $\mathcal{C}^{1,\alpha}$ defined by:

$$\mathcal{C}_{\text{nav}}^{1,\alpha} := \left\{ x = (x_1, \dots, x_d) \in \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d), x_i(t) = \int_0^t u_i(x(s), s) ds + W_i(t), \right.$$

$$\left. W_i \in \mathbb{H}^\alpha, \frac{1}{2} \leq \alpha < 1, i = 1, \dots, d \right\},$$

where u is a solution of the Navier-Stokes equation and $W = (W_1, \dots, W_d)$ satisfies

$$a_{k,l}(W(t)) = \begin{cases} -2\nu & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

- $\mathcal{C}_{\text{schr}}^\alpha(\gamma, \delta) \subset \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$ the set of $x \in \mathcal{C}^{1,\alpha}(I, \mathbb{R}^d)$ such that

$$\begin{cases} \frac{\square}{\square t} x_j(t) &= -2i\gamma \frac{\partial \ln \psi}{\partial x_j}(x(t), t), j = 1, \dots, d, \\ a_{k,k}(x(t)) &= \delta, k = 1, \dots, d, \\ a_{k,l}(x(t)) &= 0 \text{ if } k, l = 1, \dots, d, k \neq l \end{cases}$$

where $\gamma \in \mathbb{R}$ and $\delta \in \mathbb{C}$, and $\psi : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ is $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}, \mathbb{C})$.

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